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Bounds on the Capacity of Weakly Constrained Two-Dimensional Codes

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Abstract — **Upper and lower bounds are presented for the capacity of weakly constrained two-dimensional codes. The maximum entropy is calculated for two simple models of 2-D codes constraining the probability of neighboring 1s as an example. For given models of the coded data, upper and lower bounds on the capacity for 2-D channel models based on occurences of neighboring 1s are considered.**

I. Introduction

Weakly constrained codes in 1-D [1] and constrained codes in 2-D [2] have been considered. We define weakly constrained codes in 2-D, by constraining the values of the probability of subsets of N by M configurations to lie in a given interval. *Example 1.* A max. probability, p_{max}, is imposed for the occurrence of two neighboring 1's on a 2-D set of binary values.

II. Bounds on Capacities and Entropies

Let X and Y denote the stochastic variables describing the coded data written or sent and the data received, respectively. The achievable rate of the code X over the 2-D channel is given by the mutual entropy

$$I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X) \qquad (1)$$

where $H()$ and $H(|)$ denotes the entropy and conditional entropy, respectively. Maximizing $I(X;Y)$ over the code X, given the channel statistics, defines the 2-D channel capacity.

In 1-D, Y is a function of a Markov process for which the entropy $H(Y)$ may be bounded [3]. This approach also yields bounds on $H(X|Y)$. In 2-D we consider the class of fields X for which k consecutive rows may be described by a vector Markov process, e.g. Pickard Random Fields (PRF) [4], or a k-dimensional vector function of a k'-dimensional vector Markov process, eg. as in [2].

Given a vector Markov process X, $H(X)$ and $H(Y|X)$ may be calculated. The other terms of (1) may be bounded based on applying the 1-D bound [3] to the vector processes. This involves the difference of the entropies on N and $N-1$ rows and generalization of the bound to 2-D. Two examples of bounds are given below based on vector Markov processes. A lower bound based on (1) and bounding $H(X|Y)$ using an $m+l+1$ segment ($\mathbf{Y}_{t-m}^{t+l}$) of k-row vectors, $\mathbf{Y}_t$, of Y is given by

$$I(X,Y) \geq H(X) - H(X_p|\mathbf{Y}_{t-m}^{t+l}), \qquad (2)$$

where m and l are positive integers and X_p is an element of X whose position coincides with an element of $\mathbf{Y}_t$. Now let t be the index of a row by row traversal of Y. An upper bound is obtained using (1) and bounding $H(Y)$,

$$I(X,Y) \leq H(Y_t|F(Y^{t-1})) - H(Y|X), \qquad (3)$$

where $F(Y^{t-1})$ is the mapping of the causal part with respect to element Y_t, e.g. defined by a subset. $H(Y_t|F(Y^{t-1}))$ may be bounded using the 1-D result [3]. For a given weak constraint and model for X, the bounds (2-3) may be optimized over the free parameters of X.

For a given constraint defined on the probability of occurence of configurations the capacity of the code, $max\ H(X)$ may be bounded by letting a Lagrangian control the probability of a constrained configuration. As for 2-D hard constraints, a band source of width m and extending vertically is introduced defining states having $N-1$ by m elements. Each transition specifies an N by m rectangle by combining the starting and ending states. Let $H(m)$ denote the band entropy given by generating m new elements with each transition. An upper bound (on $max\ H(X)$) is given by $H(m)/m$ optimized under the given constraint. A lower bound is obtained by concatenating bands of width m now with the additional constraint that the weak constraint is still satisfied after the concatenation of independent bands.

III. Experiments

In [5], a variation of the PRF was introduced. The probabilities are derived from a 1-D binary Markov chain. For the weak constraint of Ex. 1 and given values of p_{max}, we present the values of entropy $H(X)$ optimized over the parameters of these two models. The MC-based model yielded a slightly higher entropy. Compared with upper and lower bounds calculated using the Lagrangian techniques all the values were close for the same value of p_{max}.

Let the input and output values be binary. As a simple model of the 2-D channel, related to the weak constraint of Ex. 1, we define the error probability to be a function of the neighboring 1's in the input. We present the bounds (2-3) obtained for different functions and different levels of errors, modeling X by the MC-based model [5], as iid binary data and imposing a hard constraint ($p_{max} = 0$). The optimized MC-based codes of course yields the highest capacity. The noise level has to be quite significant before the hard constraint code outperforms the iid code.

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